Extension of High-Gain Controllable Systems for Improved Accuracy

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Overview

1. Funnel-Control (FC)  
   - Prerequisites, Basic Idea -

2. System Class, Transformation & Properties  
   - Additivity of Relative Degree -

3. Nonlinear PI-Controller like Extension

4. Simulation

5. Conclusion & Outlook
Funnel-Control (FC) – Prerequisites
(developed by Ilchmann et al.)

Controller properties:
• Proportional Controller
• Adaptive Time-Varying Gain
• Control Law

\[ u(t) = \alpha \left( \partial \mathcal{F}_\phi(t), \Psi(t), \|e(t)\| \right) \cdot e(t) \]

System of Class $S$ - prerequisites:
• Relative Degree $r = 1$
• Minimum-Phase (stable zero-dynamics)
• Known High-Frequency Gain

Fig. 1: Block diagramm – System of Class $S$ controlled with FC
Funnel-Control (FC) - Basic Idea

Adaption of gain depending on (vertical) distance:

- Short distance => high gain (more aggressive control)
- Great distance => small gain (more relaxed control)

Fig. 2: Error evolution within Funnel Boundary

Time-varying (adaptive) gain $\alpha$ ($\square$) \implies$ Boundedness of control error (for sufficiently large but finite control inputs!)

$$\alpha\left(\partial F_\phi(t), \Psi(t), \|e(t)\|\right) = \frac{\Psi(t)}{d_v(t)} = \frac{\Psi(t)}{\partial F_\phi(t) - \|e(t)\|} \geq \frac{\Psi(t)}{\partial F_\phi(t)} \quad \forall t \geq 0$$

Scaling factor $\Psi(t)$ to choose minimum gain value in advance ("memory effect").
Class of nonlinear affine SISO Systems, Relative Degree

Consider the nonlinear affine (in control) SISO System $\mathcal{NL}^N_1$

$$\begin{align*}
\dot{x}(t) &= f(x) + g(x) \cdot u(t) & g(x) \neq 0 & \forall x(t) \in \mathbb{R}^N \\
y(t) &= h(x)
\end{align*}$$

of $N$-th Order with the states $x(t)$, the input $u(t)$ and the output $y(t)$.

The system has relative degree $r \geq 0$ at some point $x^*$, if the following holds true

$$L_g L_f^i h(x) = 0 \quad \forall x \text{ around } x^*; \forall i < r - 1$$

$$L_g L_f^{r-1} h(x^*) \neq 0 \quad \Rightarrow \quad y(t) \nless u(t)$$

where $L_f^i h(x) = L_f \left( L_f^{i-1} h(x) \right)$ represents recursively the i-th Lie-Derivative of the function $h(x)$ along the vector field $f$. 
Transformation in Byrnes-Isidori Normal Form (BINF)

For any known relative degree \( r \), there exists a (nonlinear) mapping

\[
z = \Phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{pmatrix}
\]

with

\[
\phi_p(x) = L_f^{p-1} h(x) \quad \forall 1 \leq p \leq r
\]

\[
\left\{ \phi_i(x) \big| L_g \phi_i(x) = 0 \right\} \quad \forall r < i \leq N
\]

which transforms the nonlinear affine system \( _1\mathcal{NL}_1 \) in BINF \( _1\mathcal{NL}_1 \)

\[
y = z_1
\]

\[
\dot{z}_p = z_{p+1} \quad \forall 1 \leq p < r
\]

\[
\dot{z}_r = b(z) + a(z) \cdot u(t) = y
\]

\[
\dot{z}_i = q_i(z) \quad \forall r < i \leq N
\]

with \( a(z) = L_g L_f^{r-1} h(\Phi^{-1}(z)) \), \( b(z) = L_f^r h(\Phi^{-1}(z)) \) and the „zero dynamics“

\[
q_i(z) = L_f \phi_i(\Phi^{-1}(z)) \quad \forall r < i \leq N.
\]
System Representation in BINF

Fig. 3: Transformed Nonlinear Affine System in BINF

Properties of the System in BINF:

- Relative degree: \( r \geq 0 \)
- Zero Dynamics: \( \dot{z}_i = q_i(z) \) for \( r+1 \leq i \leq N \)
- Sign of High-Frequency Gain (HFG): \( \rho = \text{sgn}(a(z)) \)
- Proportional Behavior for: \( \lim_{t \to \infty} b(z(t)) \neq 0 \)

(No steady state accuracy with proportional controller)
Additivity of the Relative Degree, Multiplicity of HFG

Properties of the interconnected System $\mathcal{NL}_{1}^{N} = \mathcal{NL}_{1}^{N_{1}} \ast \ldots \ast \mathcal{NL}_{1}^{N_{M}}$:

- System Order
  \[ N = \sum_{k=1}^{M} N_{k} \]

- Sign of High-Frequency Gain
  \[ \rho = \prod_{k=1}^{M} \rho_{k} = \prod_{k=1}^{M} \text{sgn}(a_{k}(z_{k})) \]

- Relative Degree
  \[ r = \sum_{k=1}^{M} r_{k} \]
Nonlinear PI-Controller like Extension

Properties of the extension with scalar functions $g'(u') > 0$ and $g_1(u_1) > 0$:

- **System Description** (with direct feedthrough)

  \[ u(t) = g'(u'(t)) \cdot u'(t) + g_1(x_1(t)) \cdot x_1(t) \]

  \[ \dot{x}_1(t) = u'(t) \]

- **Relative Degree**

  \[ r_{PI} = 0 \]

- **Sign of High-Frequency Gain** (for any fixed $x_1(t_0) = 0$)

  \[ \rho_{PI} = \text{sgn}(g'(u'(t))) > 0 \]
Auxiliary System and FC Control Loop

Properties of the auxiliary system $\mathcal{NL}_1^{\eta'} = \mathcal{NL}_1^{N_{PI}} * \mathcal{NL}_1^{N_{I}}$:

- Relative Degree
  \[ r' = r_{PI} + r_1 = r_1 = 1 \]

- Zero Dynamics (remain stable, as those of $\mathcal{NL}_1^{N_{I}}$)

- High-Frequency Gain (HFG)
  \[ \rho' = \rho_{PI} \cdot \rho_1 > 0 \]

- Integrating Behavior
  \[ \lim_{t \to \infty} b'(z, x_I) = \lim_{t \to \infty} (b(z) + a(z) g_I(x_I) x_I) = 0 \]

Fig. 6: Nonlinear System with Extension (Auxiliary System) and FC Loop
Simulation Results – PT1 with FC and FC+PI (with Noise)

**Fig. 8a:** Output $y(t)$ for FC and FC+PI

**Fig. 8b:** Error $e(t)$ for FC and FC+PI, Funnel $F(t)$

**System and Controller:**

$$F_{PT_1}(s) = \frac{V_s}{1 + sT_s}$$

$$F_{PI}(s) = K_P \left( 1 + \frac{1}{sT_l} \right)$$

**Funnel Boundary and Scaling Function:**

$$\partial F_{\exp}(t) = \varphi_{\exp,0} \cdot \exp\left(-\frac{t}{T_{\exp}}\right) + \varphi_{\exp,\infty}$$

$$\Psi(t) = \partial F_{\exp}(t)$$
Consider the nonlinear affine example SISO System in BINF

\[
\begin{align*}
\dot{y} = \dot{z}_1 &= \tanh(z_1) + z_1^2 + z_2^3 + \sqrt{z_1^2 + |z_2|} + 1 \cdot u \\
\dot{z}_2 &= -\sin(z_1) - z_2 \exp(z_2^2) \\
y &= z_1
\end{align*}
\]

of 2-th Order with the states \(x(t)\), the input \(u(t)\) and the output \(y(t)\).

The system fulfills requirements for FC with

- Relative degree

\[
\dot{y} \nless\nless u \quad \Rightarrow \quad r = 1
\]

- Stable Zero-Dynamics (\(\dot{y} = y = 0\))

\[
V_q(z) = z_2^2 \quad \Rightarrow \quad \dot{V}_q(z) = 2z_2 \dot{z}_2 = -2z_2^2 \exp(z_2^2) < 0 \quad \forall z
\]

- Positive (known) Sign of High-Frequency Gain

\[
\rho = \text{sgn}\left(\sqrt{z_1^2 + |z_2|} + 1\right) > 0
\]
Simulation Results – Nonlinear System with FC and FC+PI

**Fig. 9a: Output y(t) for FC and FC+PI**

**System and Controller:**
\[
\dot{y} = \dot{z}_1 = b(z) + a(z) \cdot u(t)
\]
\[
y = z_1
\]
\[
F_{PI}(s) = K_P \left(1 + \frac{1}{sT_I}\right)
\]

**Fig. 9b: Error e(t) for FC and FC+PI, Funnel F(t)**

**Funnel Boundary and Scaling Function:**
\[
\partial F_{exp}(t) = \varphi_{exp,0} \cdot \exp\left(-\frac{t}{T_{exp}}\right) + \varphi_{exp,\infty}
\]
\[
\Psi(t) = \partial F_{exp}(t)
\]
Conclusion:

• Steady state accuracy (despite unknown plant!)
• Robustness:
  Unknown parameters & nonlinearities do *not* affect control performance
• Measurement noise is admissible
• No Identification/observation and compensation of nonlinearities necessary

Outlook:

• Mathematical results applicable to nonlinear multi-mass flexible servo system
  (Presentation H. Schuster in „Adaptive Control II“ ThB10.1)
• Further generalization of the extension structure