These notes discuss the content of the lecture of the summer term 2012 in brief form. A more detailed and comprehensive treatment of the material is given in the script

**Non-identifier based adaptive control in mechatronics**
— Part I: speed control —

All references (to Definitions, Lemmas or Theorems, etc.) refer to this script.

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1 Non-identifier based adaptive control in mechatronics: An introduction (2012/04/20)

1.1 Learning outcomes

After this course you should be able to

- evaluate and create models of (standard) mechatronic systems including (dynamic) friction
- understand, analyze and evaluate (nonlinear, functional) differential equations (e.g. existence and uniqueness of a solution)
- understand and apply mathematical tools (e.g. Lyapunov’s 2. (direct) method, proof by contradiction/induction, ...) to prove theoretical results
- create and design robust non-identifier based adaptive controllers and internal models for speed (and position) control of mechatronic systems

1.2 Motivation: Reference tracking under load (CNC-turning machine)

- problem statement
  - reference trajectory \((\Omega_{\text{ref}}, x_{\text{ref}}, y_{\text{ref}}): \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^3\) given
  - high-precision position and speed control, e.g. for prescribed \(\lambda > 0\)
    \[
    \forall t \geq t_0: |e(t)| = |y_{\text{ref}}(t) - y(t)| \leq \lambda.
    \]

- challenges
  - nonlinear effects (e.g. actuator saturation, friction)
  - friction and disturbances unknown and varying
  - system parameters (only) roughly known
1.3 Standard solution(s) in industry

- **heuristic or empirical** controller design ⇒ ‘trial-and-error’ (time consuming)
- **modell-based** controller design ⇒ system identification (complex/time consuming)
- **general challenge** ⇒ desired tracking accuracy *not explicitly* included in design

\[
\forall t \geq t_0: |y(t) - y(t) - y(t)| \leq \lambda
\]

1.4 Motivation for non-identifier based adaptive control

- **Motivation 1**: desirable to have tools at hand which directly allow to account for desired tracking accuracy, e.g.

- **Motivation 2**: to avoid stick-slip (major problem due to friction), the simplest approach is high gain in the feedback ("stiff" closed-loop system).

- **Motivation 3**: mechatronic systems (at least of low order) are “structurally” known:
  - the “direction of influence” of control input on system output (i.e. the sign of the “high-frequency gain” of the system)
– the time derivative (e.g. \( \frac{d^2}{dt^2} y(t) \)) of the system output which is directly affected by the control input (i.e. the “relative degree” of the system).
– the “internal dynamics” of the system are “input-to-state stable” (i.e. stability of the “zero-dynamics” or, for linear systems, the “minimum-phase” property).

Non-identifier based (high-gain) adaptive control allows to

- think in “system classes” (robustness: often only the signs of the system parameters must be known)
- prescribe desired accuracy a priori (e.g. according to customer specifications)
- reduce friction effects

### 1.5 Direct and indirect adaptive control

![Diagram of adaptive control](image)

**Figure 1: Indirect adaptive control**: The adaptive controller estimates/identifies the system. Controller parameters are adjusted according to the system estimate.
Non-identifier based adaptive control in mechatronics

Typical non-identifier based adaptive controllers are given by:

1. Classical high-gain adaptive control

\[
    u(t) = k(t) \cdot \left( y_{\text{ref}}(t) - y(t) \right)
\]

\[
    \dot{k}(t) = |y_{\text{ref}}(t) - y(t)|^2, \quad k(0) = k_0 \in \mathbb{R}
\]

2. Adaptive \( \lambda \)-tracking control

\[
    u(t) = k(t) \cdot \left( y_{\text{ref}}(t) - y(t) \right)
\]

\[
    \dot{k}(t) = \max\{|e(t)| - \lambda, 0\}, \quad k(0) = k_0 \in \mathbb{R}, \quad \lambda > 0
\]

where bounded noise (see Fig. 3)

\[
    n_m(\cdot) \in W^{1,\infty}(\mathbb{R}_\geq 0; \mathbb{R}) \quad \Rightarrow \quad \forall t \geq 0 \exists c > 0: |n_m(t)| \leq c
\]

should be considered appropriately: choose \( \lambda \) large enough:

\[
    \Rightarrow \lambda \gg c
\]

3. Funnel control (with positive, Lipschitz continuous funnel boundary \( \psi(\cdot) \))

\[
    u(t) = k(t) \cdot \left( y_{\text{ref}}(t) - y(t) \right) \quad \text{where} \quad k(t) = \frac{1}{\psi(t) - |e(t)|} \quad \text{and} \quad \psi : \mathbb{R}_\geq 0 \to [\lambda, \infty)
\]
First motivating examples of the application of non-identifier based adaptive controllers are discussed in Tutorial 1 (see Tutorial notes).

### 1.6 Modeling of standard mechatronic systems

#### 1.6.1 Components of mechatronic systems

For more details on the modeling of the components of standard mechatronic systems see script (p. 10-19).
1.6.2 Stiff servo-system (1MS)

The mathematical model of the saturated 1MS is given by

\[
\begin{align*}
\frac{d}{dt} \mathbf{x}(t) &= A \mathbf{x}(t) + b \text{sat}_{\hat{u}_A}(u(t) + u_A(t)) + B_L \left( m_L(t) + (\mathbf{F}_1 \omega)(t) \right), \\
y(t) &= c^\top \mathbf{x}(t), \\
x(0) &= x^0 \in \mathbb{R}^2
\end{align*}
\]  

(1MS)

where system matrix \( A \) (viscous friction terms are included), input vector \( b \), disturbance input matrix \( B_L \), output vector \( c \), disturbances \( u_A(\cdot), m_L(\cdot) \), friction operators \( \mathbf{F}_1, \mathbf{F}_2 \) and system parameters are as follow

\[
\begin{align*}
A &= \begin{bmatrix}
-\frac{\nu_1 + \nu_2/g^2}{\Theta} & 0 \\
-\frac{1}{\Theta} & 0
\end{bmatrix}, \\
b &= \left( \frac{k_A \Theta}{\Theta} \right), \\
B_L &= \begin{bmatrix}
-\frac{1}{\Theta} & -\frac{1}{g^2 \Theta} \\
0 & 0
\end{bmatrix}, \\
c &= \mathbb{R}^2, \Theta > 0, \\
g_r &\in \mathbb{R} \setminus \{0\}, \nu_1, \nu_2 > 0, \hat{u}_A, k_A > 0, u_A(\cdot), m_L(\cdot) \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \text{ and } \forall i \in \{1, 2\}: \\
\mathbf{F}_i : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \text{ with } M_{\mathbf{F}_i} := \sup \{|(\mathbf{F}_i \omega)(t)| \mid t \geq 0, \omega(\cdot) \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})\} < \infty.
\end{align*}
\]  

(1MS-Data)

The state variable \( \mathbf{x}(t) = (\omega(t), \phi(t))^\top \) represents angular velocity (speed) and angle (position) at time \( t \geq 0 \) [s] in [rad/s] and [rad], respectively. The symbols have the following physical meaning [dimension]:

- actuator \( u_A(\cdot) \) [Nm] and load disturbance \( m_L(\cdot) \) [Nm]
- actuator gain \( k_A > 0 \) [1]
- inertia \( \Theta > 0 \) [kg m]
- gear ratio \( g_r \in \mathbb{R} \setminus \{0\} \) [1] (mostly known)
- viscous friction coefficients \( \nu_1, \nu_2 \geq 0 \) \( \frac{\text{Nms}}{\text{rad}} \)
- measurement error(s) or noise \( n_m(\cdot) \) [rad] (and \( \dot{n}_m(\cdot) \) [rad/s])

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In contrast to many text books, friction is modeled as nonlinear, dynamic operator, which here implies that friction is not continuous (see Fig. 4). For simplicity, in the following, we consider the nonlinear friction operator $F_1 \omega$ as nonlinear, locally Lipschitz continuous function, e.g. $\omega \mapsto \text{atan}(\omega)$. More details on nonlinear, dynamic friction modeling can be found in [1, Section 1.4.5].

Figure 4: Nonlinear dynamic but bounded friction behavior $F_1 \omega$ (e.g. due to dynamic LuGre friction model) with linear, but unbounded viscous friction $\nu_1 \omega$: friction is a continuous function of speed $\omega$!
2 High-gain adaptive stabilization (2012/04/27)

2.1 Motivation: ‘Structural properties’ of (1MS) (speed control)?

For speed control, the stiff servo system (1MS) simplifies to (without noise \( \dot{n}_m(\cdot) \) and friction operator \( \mathfrak{F}_1, \mathfrak{F}_2 \))

\[
\begin{align*}
\dot{\omega}(t) &= -\nu_1 + \nu_2/\Theta \omega(t) + \frac{k_A}{\Theta} \text{sat}_{\hat{\alpha}}(u(t) + u_A(t)) - \frac{f_1(\omega(t))}{\Theta} - \frac{m_L(t) + f_2(\omega(t))}{g r}, \\
y(t) &= c_1 \omega(t),
\end{align*}
\]

(1MS\textsuperscript{ω})

where

\[\Theta > 0, \ g_r \in \mathbb{R} \setminus \{0\}, \ \text{sign}(g_r) \text{ known}, \ \nu_1, \nu_2 \geq 0, \ \hat{u}_A > 0, \ u_A(\cdot), \ m_L(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}), \ c_1 \in \{1, 1/g_r\} \text{and } f_1(\cdot), f_2(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \text{ and locally Lipschitz continuous.}
\]

Structural properties of (1MS\textsuperscript{ω}) with (1MS\textsuperscript{ω}-Data):

- relative degree (see Definition 2.1)

\[
\dot{y}(t) = c_1 \dot{\omega}(t) = \ldots + c_1 \frac{k_A}{\Theta} \text{sat}_{\hat{\alpha}}(u(t) + u_A(t)) + \ldots \implies r = 1.
\]

- high frequency gain (see Definition 2.4)

\[
\gamma_0 := c_1 \frac{k_A}{\Theta} \text{ where } \text{sign}(\gamma_0) \text{ is known.}
\]

- minimum-phase? (see Definition 2.7)

\[n = 1 \implies r = 1 \implies \text{No internal dynamics, hence minimum-phase.}
\]

2.2 High-gain adaptive stabilization

2.2.1 Considered system class (relative-degree-one case)

Definition 2.1 (System class \( S^{\text{lin}}_1 \)). A system of form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t), \quad n \in \mathbb{N}, \ x(0) = x^0 \in \mathbb{R}^n, \\
y(t) &= c^T x(t)
\end{align*}
\]

(2.1)

is of Class \( S^{\text{lin}}_1 \) if, and only if, the following system properties (sp) hold:

- \((S^{\text{lin}}_1-\text{sp}_1)\) relative degree is one and sign of the high-frequency gain is known, i.e.

\[
\gamma_0 := c^T b \neq 0 \quad \text{and} \quad \text{sign}(\gamma_0) \text{ is known;}
\]

- \((S^{\text{lin}}_1-\text{sp}_2)\) it is minimum-phase, i.e. \( \forall s \in \mathbb{C}_{\geq 0}: \det \left[ sI_n - A \ c^T \ b \right] \neq 0 \), and

- \((S^{\text{lin}}_1-\text{sp}_3)\) the (regulated) output \( y(\cdot) \) is available for feedback.
Goal: design single adaptive controller to stabilize any system of $S_i^{\text{lin}}!$

2.2.2 Motivation: Root locus

controller

\[ u = -ky \]

system

\[ F_1(s) = \frac{(s+4)(s+5)}{(s-1)^2(s+1)} \]

‘Structural properties’ of $F_1(s)$:

- relative degree (pole excess): $r = 1$
- positive high-frequency gain ($\lim_{s \to \infty} sF_1(s) = 1$)
- minimum-phase (numerator is Hurwitz)

\[ \Rightarrow \exists k^* > 0 \text{ such that closed-loop is stable for all } k > k^*, \text{ however } k^* \text{ clearly depends on system data and must be known a priori to design a stabilizing controller. Questions} \]

- can this minimum gain $k^*$ found ‘online’ by some suitable (gain) adaption law?
- how to achieve $k(t^*) > k^*$ for some $t^* \geq 0$?
- does the adapted gain $k(\cdot)$ remain bounded?
2.2.3 Our first result

**Theorem 2.2** (High-gain adaptive control for a simple system). Consider the following system given by

\[
\begin{align*}
\dot{y}(t) &= a_1 y(t) + a_2 z(t) + \gamma_0 u(t), \\
\dot{z}(t) &= a_3 y(t) - a_4 z(t)
\end{align*}
\]

\( (y(0), z(0)) = (y_0, z_0) \in \mathbb{R}^2, \quad a_1, a_2, a_3 \in \mathbb{R}, \quad a_4, \gamma_0 \in \mathbb{R}_{>0}. \) \hfill (2.2)

The high-gain adaptive controller

\[ u(t) = -k(t) y(t) \quad \text{where} \quad \dot{k}(t) = y(t)^2, \quad k(0) = k_0 \]  \hfill (HG_1)

with design parameter \( k_0 > 0 \) applied to (2.2) yields a closed-loop initial-value problem with the following properties:

(i) there exists a unique, maximal solution \( (y, z, k) : [0, T) \mapsto \mathbb{R}^2 \times \mathbb{R}_{>0}, T \in (0, \infty] \);

(ii) the solution is global, i.e. \( T = \infty \);

(iii) all signals are bounded, i.e. \( y(\cdot), z(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \) and \( k(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}_{>0}) \);

(iv) \( \lim_{t \to \infty} \dot{k}(t) = 0 \) and \( \lim_{t \to \infty} (y(t), z(t)) = 0_2 \).

Before we can show our first result, we need to understand the concept of solution(s) of ordinary differential equations (ODEs).

2.2.4 Existence and uniqueness of solutions

Consider the initial-value problem (IVP) given by ordinary differential equation

\[ \dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \]  \hfill (IVP)

where

- \( f : \mathcal{I} \times \mathcal{D} \to \mathbb{R}^n \), \( n \in \mathbb{N} \), is called the right-hand side of (IVP) with open (time) interval \( \mathcal{I} \subseteq \mathbb{R} \) and open, non-empty (state) domain \( \mathcal{D} \subseteq \mathbb{R}^n \)

- \( t_0 \in \mathcal{I} \) is the initial time and

- \( x_0 \in \mathcal{D} \) is the initial state (or condition).

Questions:

- what is a solution of the initial-value problem (IVP)?

- do solutions (always) exist?
is there a unique solution?

- does solution exist on complete interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$?

**Answers:**

- First answer: A continuously differentiable function $x: [t_0, T) \rightarrow \mathbb{R}^n$, $T = T(t_0, x_0) \in (t_0, \infty)$ is called solution of the initial-value problem (IVP) if it satisfies (IVP) for all $t \in [t_0, T) \subseteq \mathcal{I}$ and $x(t_0) = x_0$.

- Second answer: there exist differential equations which do not have a solution (in the sense as stated above, see motivating examples below)

- Third answer: there does not necessarily exist a unique solution (see motivating examples below)

- Fourth answer: there does not necessarily exist a global solution (see motivating examples below)

**Discussion of motivating examples:**

**Example 2.3.**

\[
\dot{x}(t) = \begin{cases} 
-1, & x(t) \geq 0 \\
1, & x(t) < 0
\end{cases}, \quad x(0) = 0
\]

$\implies$ no solution, see Fig. 6 (recall also the practical simulation exercise).

**Example 2.4.**

\[
\dot{x}(t) = 2\sqrt{|x(t)|}, \quad x(0) = 0
\]

Try to find a solution:

$\implies x_1(t) = 0 \forall t \geq 0$

$\implies \alpha < 0 < \beta$

\[
x_2(t) = \begin{cases} 
-(\alpha - t)^2, & t \in (-\infty, \alpha] \\
0, & t \in (\alpha, \beta) \\
(t - \beta)^2, & t \in [\beta, \infty)
\end{cases}
\]

$\implies$ no unique solution.

**Example 2.5.**

\[
\dot{x}(t) = x(t)^2, \quad x(0) = 1
\]
A solution is given by $x(t) = \frac{1}{1-t}$. We verify this claim:

- **initial value:** $x(0) = \frac{1}{1} = 1 \implies OK$

- **Solution of IVP:** $\frac{d}{dt} x(t) = \frac{d}{dt} (1-t)^{-1} = 1 \cdot \frac{1}{(1-t)^2}$ for all $t \in [0, 1) \implies OK$

- **maximal time of existence:** $T = T(1) = 1 \implies$ solution not a global!

**Example 2.6.**

\[
\dot{x}(t) = -\left(1 + \frac{1}{t}\right) x(t) + \frac{x(t)^3}{t^2}, \quad x(t_0) = x_0 \in \mathbb{R}, \quad t_0 > 0
\]

\[
x(t) = \frac{x_0 t_0}{t\sqrt{x_0^2 t_0^2 - (x_0^2 t_0^2 - 1) \exp(2(t - t_0))}}
\]

Possible singularity in the right-hand side at:

\[
x_0^2 t_0^2 = (x_0^2 t_0^2 - 1) \exp(2(t - t_0)) \implies \ln \left(\frac{x_0^2 t_0^2}{x_0^2 t_0^2 - 1}\right) = 2(t - t_0)
\]

For $x_0^2 t_0^2 > 1$, the solution explodes and $T = T(t_0, x_0) = t_0 + \ln \left(\frac{x_0^2 t_0^2}{\sqrt{x_0^2 t_0^2 - 1}}\right) < \infty$ (finite escape time).

A solution of the initial-value problem (IVP) exists (Peano existence theorem) and it is unique (Picard-Lindelöf theorem, see Theorem D.32) if the right-hand side of (IVP) satisfies two pre-suppositions (see Definition D.31).
3 Proof of our first result (2012/05/04)

Proof of Theorem 2.2. The closed-loop system is given by

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} y(t) \\ z(t) \\ k(t) \end{pmatrix} &= \begin{pmatrix} (a_1 - \gamma_0 k(t))y(t) + a_2 z(t) \\ a_3 y(t) - a_4 z(t) \\ y(t)^2 \end{pmatrix}, \\
\begin{pmatrix} y(0) \\ z(0) \\ k(0) \end{pmatrix} &= \begin{pmatrix} y_0 \\ z_0 \\ k_0 \end{pmatrix} \in \mathbb{R}^3 \\
\end{align*}
\]

(3.3)

Note that \( f \) does not explicitly dependent on \( t \).

Step 1: We show that Assertion (i) holds true. Define

\[
I = \mathbb{R}_{\geq 0} \quad \text{and} \quad \mathcal{D} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{\geq 0} 
\]

The right-hand side of the closed loop system (3.3) is

i) continuous on \( I \times \mathcal{D} \)

ii) and, for any compact set (here a ball, see Fig. 7) \( \mathcal{C} \subset \mathcal{D} \), the following hold:

\[ (\text{hint: } y + \tilde{y} \leq |y| + |\tilde{y}|) \]

Figure 7: Compact ball in \( \mathbb{R}^3 \).
Hence,
\[
\left\| f(y, z, k) - f(\tilde{y}, \tilde{z}, \tilde{k}) \right\| \leq |a_1| \cdot |y - \tilde{y}| + |a_2| \cdot |z - \tilde{z}| + |\gamma_0| \cdot |ky - \tilde{ky}| + |a_3| \cdot |y - \tilde{y}| + |a_4| \cdot |z - \tilde{z}| + |y^2 - \tilde{y}^2| \leq (|a_1| + |\gamma_0|M_\varepsilon + |a_3| + 2M_\varepsilon)|y - \tilde{y}| + (|a_2| + |a_4|)|z - \tilde{z}| + |\gamma_0|M_\varepsilon|k - \tilde{k}| \leq L_\varepsilon \cdot \left\| \begin{pmatrix} y \\ z \\ k \end{pmatrix} - \begin{pmatrix} \tilde{y} \\ \tilde{z} \\ \tilde{k} \end{pmatrix} \right\| \Rightarrow OK
\]

where \( L_\varepsilon = c_0 + c_1 + c_2 \Rightarrow f(\cdot, \cdot, \cdot) \) is locally Lipschitz continuous.
Therefore, in view of Theorem D.32, there exists a unique solution \( x = (y, z, k) : [0, T) \rightarrow \mathbb{R}^3 \) with maximal \( T \in (0, \infty] \). This completes step 1 of the proof and shows assertion (i) of Theorem 2.2.

Step 2: We show some technical inequalities:

Clearly,
\[
\forall m > 0, \forall a, b \in \mathbb{R} : \quad \pm 2ab = -\left( \frac{a}{\sqrt{m}} \mp \sqrt{mb} \right)^2 + \frac{a^2}{m} + mb^2 \leq \frac{a^2}{m} + mb^2 \quad \text{(BF)}
\]

and, for any \( a_4 > 0 \), introduce the following Lyapunov candidate:
\[
V : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ : \quad \left( \begin{array}{c} y \\ z \\ k \end{array} \right) \Rightarrow V(y, z) = \frac{1}{2}y^2 + \frac{1}{2a_4}z^2
\]
\[
\forall (y, z) \in \mathbb{R}^2 : \quad 0 \leq \frac{1}{2} \min\{1, \frac{1}{a_4}\} \left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|^2 \leq V(y, z) \leq \frac{1}{2} \max\{1, \frac{1}{a_4}\} \left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|^2
\]

From Step 1 we know, that there exists a solution \( x(\cdot) = (y(\cdot), z(\cdot), k(\cdot)) \) on \([0, T)\), hence we
may differentiate $V(y(\cdot), z(\cdot))$ along this solution, i.e.

$$
\forall t \in [0, T) : \\
\dot{V}(y(t), z(t)) = y(t)\dot{y}(t) + \frac{1}{a_4}z(t)\dot{z}(t)
$$

$$
= y(t)\left((a_1 - \gamma_0 k(t))y(t) + a_2 z(t)\right) + \frac{1}{a_4}z(t)\left(a_3 y(t) - a_4 z(t)\right)
$$

$$
\leq - (\gamma_0 k(t) - |a_1|) y(t)^2 + \left(|a_2| + \frac{|a_3|}{a_4}\right) |y(t)| \cdot |z(t)| - z(t)^2
$$

$$
\leq - (\gamma_0 k(t) - |a_1|) y(t)^2 + \frac{1}{2} \left(|a_2| + \frac{|a_3|}{a_4}\right)^2 y(t)^2 - \frac{1}{2} z(t)^2
$$

$$
\leq - (\gamma_0 k(t) - |a_1|) y(t)^2 + \frac{1}{2m} \left(|a_2| + \frac{|a_3|}{a_4}\right)^2 + \frac{m}{2} z(t)^2 - z(t)^2.
$$

We choose $m = 1$ (arbitrary) in (BF), then

$$
\dot{V}(y(t), z(t)) \leq - \left(\gamma_0 k(t) - |a_1| - \frac{1}{2} \cdot \frac{1}{1} \left(|a_2| + \frac{|a_3|}{a_4}\right)^2\right) y(t)^2 - \frac{1}{2} z(t)^2
$$

high gain: term will eventually get smaller than zero

Step 3: We show that $k(\cdot)$ is bounded on $[0, T)$:

Proof by contradiction: Assume that $k(\cdot)$ is unbounded on $[0, T)$. Since

$$
\forall t \in [0, T): \quad \dot{k}(\cdot) \geq 0 \quad \text{(non-decreasing)}
$$

therefore

$$
\exists t^* \geq 0 \forall t \in [t^*, T]: \quad k(t) \geq k^* := \frac{1}{\gamma_0} \left(|a_1| + \frac{1}{2} \left(|a_2| + \frac{|a_3|^2}{|a_4|}\right) + \frac{1}{2}\right).
$$
Hence,\
\[
\forall t \in [t^*, T) : \quad \dot{V}(y(t), z(t)) = \frac{d}{dt} \xi(t) \leq -\frac{1}{2}y(t)^2 - \frac{1}{2}z(t)^2 \leq -\frac{1}{2\mu_2} \cdot V(y(t), z(t)) =: \xi(t) - \frac{1}{2} \left\| \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \right\|^2
\]
where \(\mu_2 := \frac{1}{\max(1, \frac{1}{\tau})}\).

From the Bellman-Gronwall Lemma (in its differential form, see Lemma E.46), we conclude that\
\[
\forall t \in [t^*, T) : \quad V(y(t), z(t)) \leq V(y(t^*), z(t^*)) \cdot \exp(-2\mu_2(t - t^*)) \tag{3.4}
\]
and therefore\
\[
\forall t \in [0, T) : \quad k(t) = k(t^*) + \int_{t^*}^{t} k(\tau) d\tau = k(t^*) + \int_{t^*}^{t} \frac{1}{\mu_1} V(y(\tau), z(\tau)) d\tau \leq k(t^*) + \int_{t^*}^{t} \frac{1}{\mu_1} V(y(t^*), z(t^*)) \cdot \int_{t^*}^{\tau} \exp(-2\mu_2(\tau - t^*)) d\tau < \infty,
\]
which implies that \(k(\cdot)\) is bounded on \([0, T)\).

**Step 4:** We show that Assertion (ii) holds true, i.e. \(T = \infty\).
Since \(k(\cdot)\) is non-decreasing and bounded on \([0, T)\), the limit \(\lim_{t \to \infty} k(t) = k_\infty < \infty\) exists. Hence, we can find an upper bound for the right-hand side of (3.3) as follows
\[
\forall t \in [0, T) : \quad \frac{d}{dt} \left\| \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \right\| \leq \sigma \left\| \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \right\| \text{ where } \sigma > 0
\]
Invoking the Bellman-Gronwall Lemma (in its differential form, see Lemma E.46) again yields
\[
\left\| \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} \right\| \cdot e^{\sigma t} \quad \Longrightarrow \quad T = \infty.
\]

**Step 5:** We show that Assertion (iii) and (iv) hold true.
From Steps 3 and 4, it follows that
\[
k(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}_{> 0}).
\]
Moreover, since
\[
\int_{0}^{\infty} \frac{y(\tau)^2}{k(\tau)} d\tau \leq k_\infty - k_0 < \infty \quad \Longrightarrow \quad y(\cdot) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}; \mathbb{R}) \quad \text{(but } \not\Rightarrow y(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})\text{)}.
\]
We look at the internal dynamics \(\dot{z}(t) = a_3 y(t) - a_4 z(t)\) with solution (derived by ‘variation-of-
constants')

\[ \forall t \geq 0: \quad z(t) = z_0 e^{-at} + \int_0^t e^{-a(t-\tau)} \cdot y(\tau) \cdot h(t-\tau) \, d\tau \quad \Rightarrow \quad z(\cdot) \in L^2(\mathbb{R}_{\geq 0}; \mathbb{R}). \]

Combining the results above yields

\[ \frac{d}{dt} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = f(y(t), z(t), k(t)) \quad \Rightarrow \quad \frac{d}{dt} \begin{pmatrix} y(\cdot) \\ z(\cdot) \end{pmatrix} \in L^2(\mathbb{R}_{\geq 0}, \mathbb{R}^2). \]

We want to show that \( y(\cdot), z(\cdot) \) are bounded. Since \( y(\cdot), z(\cdot) \in L^2(\mathbb{R}_{\geq 0}; \mathbb{R}) \) and \( \dot{y}(\cdot), \dot{z}(\cdot) \in L^2(\mathbb{R}_{\geq 0}; \mathbb{R}) \), we can apply Lemma E.44:

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t) = 0 \]
\[ \lim_{t \to \infty} \dot{k}(t) = \lim_{t \to \infty} y(t)^2 = 0 \]

\[ \Rightarrow \quad y(\cdot), z(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}), \]

which completes the proof. \( \square \)
4 Adaptive $\lambda$-tracking control and funnel control: An introduction (2012/06/01)

4.1 High-gain adaptive stabilization

One can show the following theorem:

**Theorem** (High-gain adaptive control for systems of class $S_{\text{lin}}^1$ (see Theorem 2.21)). Consider a system of class $S_{\text{lin}}^1$ given by

$$\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t), \\
y(t) &= c^T x(t)
\end{align*} \quad \begin{array}{c}
n \in \mathbb{N}, \ x(0) = x^0 \in \mathbb{R}^n, \\
(A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^n.
\end{array} \quad (\text{SYS}_{\text{lin}}^1)
$$

The high-gain adaptive controller

$$u(t) = -\text{sign}(c^T b) k(t) y(t) \quad \text{where} \quad \dot{k}(t) = q_1 |y(t)|^{q_2}, \quad k(0) = k_0 \quad (\text{HG}_1)$$

with design parameters $q_1 > 0$, $q_2 \geq 1$ and $k_0 > 0$ applied to (SYS$_{\text{lin}}^1$) yields a closed-loop initial-value problem with the following properties:

(i) there exists a unique, maximal solution $(x, k) : [0, T) \to \mathbb{R}^n \times \mathbb{R}_{>0}$, $T \in (0, \infty]$;

(ii) the solution is global, i.e. $T = \infty$;

(iii) all signals are bounded, i.e. $x(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$ and $k(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}_{>0})$;

(iv) $\lim_{t \to \infty} \dot{k}(t) = 0$ and $\lim_{t \to \infty} x(t) = 0_n$.

**Proof.** Proof was not explicitly presented (see Proof of Theorem 2.21 on p. 46-50. It is quite similar to the proof of "our first result").

**Drawback of this result:**

Already the simple nonlinear stiff-servo system (1MS$^\omega$) with (1MS$^\omega$-Data) is not element of class $S_{\text{lin}}^1$ (why?). Hence, a wider system class and/or more robust controllers are needed ...

4.2 Motivation for wider system class & more robust controllers

- **Motivation 1:** System class $S_{\text{lin}}^1$ is restrictive (e.g. (1MS$^\omega$) $\notin S_{\text{lin}}^1$)
  - linear
  - no (external) disturbances (possibly piecewise continuous!)
  - no (nonlinear) perturbations

- **Motivation 2:** high-gain adaptive stabilizing controller not robust (gain drift due to noisy measurements or external disturbances)

- **Motivation 3:** reference tracking not considered (only stabilization, or for certain reference classes, asymptotic tracking with internal models)
Gain drift

(a) due to noise $n_m(\cdot) \in \mathcal{W}^{2,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ ($u_d(\cdot) = 0$): $-k(\cdot)$, $y(\cdot)$.

(b) due to disturbance $u_d(\cdot) \in \mathcal{L}^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ ($n_m(\cdot) = 0$): $k(\cdot)$ and $y(\cdot)$.

Figure 9: Simulation results for closed-loop system (4.5), (4.6) (only shown for the first 20 [s]).

Simulations are performed in Matlab/Simulink for the following system

$$
\begin{align*}
\dot{x}(t) &= 10x(t) + (u(t) + u_d(t)) \\
y(t) &= x(t) + n_m(t)
\end{align*}
$$

and controller

$$
u(t) = -k(t)y(t) \quad \text{where} \quad \dot{k}(t) = y(t)^2, \quad k(0) = 1.
$$

4.3 Adaptive $\lambda$-tracking control

The adaptive $\lambda$-tracking controller assures

- tracking with prescribed asymptotic accuracy, i.e. for all $\lambda > 0$, the tracking error $e(t) = y_{ref}(t) - y(t)$ asymptotically converges into the “$\lambda$-strip”, i.e.

$$
\lim_{t \to \infty} \text{dist} \left( |e(t)|, [0, \lambda] \right) = 0
$$

- state variable is bounded, i.e. $x(\cdot) \in \mathcal{L}^{\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$;
control action is bounded, i.e. $u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R})$.

4.3.1 Application: Adaptive $\lambda$-tracking speed control of unsaturated ($1MS^\omega$)

**Theorem 4.1** (Adaptive $\lambda$-tracking speed control of unsat. ($1MS^\omega$)). Consider the mechatronic system ($1MS^\omega$) with ($1MS^\omega$-Data) and $\dot{u}_A = \infty$ (unsaturated actuator). Then, for arbitrary initial value $\omega_0 \in \mathbb{R}$ and reference signal $y_{ref}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_\geq 0; \mathbb{R})$, the adaptive $\lambda$-tracking controller

$$
\begin{aligned}
u(t) &= \text{sign}(g_r) k(t)e(t), \quad \text{where} \quad e(t) = y_{ref}(t) - y(t) \\
\dot{k}(t) &= d_\lambda(|e(t)|)^2, \quad k(0) = k_0
\end{aligned}
$$

with design parameters $k_0 > 0$ and $\lambda > 0$ applied to ($1MS^\omega$) yields a closed-loop initial-value problem with the properties:

(i) there exists a unique, maximal solution $(\omega, k) : [0, T) \to \mathbb{R} \times \mathbb{R}_\geq 0$, $T \in (0, \infty]$;

(ii) the solution is global, i.e. $T = \infty$;

(iii) all signals are bounded, i.e. $\omega(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R})$ and $k(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R}_\geq 0)$;

(iv) the $\lambda$-strip is asymptotically reached, i.e. $\lim_{t \to \infty} \text{dist}\left(|e(t)|, [0, \lambda]\right) = 0$.

**Proof.** The proof was not explicitly presented. It is a special case of the proof of Theorem 2.27 (see p. 56-62).

4.4 Funnel control

The funnel controller assures

- tracking with prescribed transient accuracy, i.e.

$$\forall \lambda > 0 \forall \psi(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_\geq 0; [\lambda, \infty)) \forall |e(0)| < \psi(0) \forall t \geq 0: |e(t)| < \psi(t);$$

- state variable is bounded, i.e. $x(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R}^n)$;

- control action is bounded, i.e. $u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R})$. 

The funnel controller assures

- tracking with prescribed transient accuracy, i.e.

$$\forall \lambda > 0 \forall \psi(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_\geq 0; [\lambda, \infty)) \forall |e(0)| < \psi(0) \forall t \geq 0: |e(t)| < \psi(t);$$

- state variable is bounded, i.e. $x(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R}^n)$;

- control action is bounded, i.e. $u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_\geq 0; \mathbb{R})$. 

---

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4.4.1 Motivation for funnel control

- **Motivation 1:** Adaptive $\lambda$-tracking (and high-gain adaptive) control exhibit a non-decreasing gain and so, as time tends to infinity, it is likely that e.g. noise sensitivity permanently exceeds an acceptable level (if gain adaption is not stopped).

- **Motivation 2:** Adaptive $\lambda$-tracking control assures tracking with prescribed asymptotic accuracy, however statements on the transient accuracy are not possible; e.g.
  - albeit bounded large overshoots might occur and
  - the $\lambda$-strip is not reached in finite time (in general)

- **Motivation 3:** Input saturations are not yet considered (e.g. actuator saturation of electrical drives).

Unpredictable overshoots and “infinite transient time”

![Graph](image1)

**Figure 10:** Simulation results for closed-loop system (4.5), (4.7) with $y_{\text{ref}}(\cdot) = 10$, $\lambda = k(0) = q_1 = 1$, $q_2 = 2$, $y(0) = 0$, noise $n_{\text{m}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}; [-0.1, 0.1])$ and disturbance $u_{d}(\cdot) = 0$.

Simulations are performed in Matlab/Simulink for system (4.5) and adaptive $\lambda$-tracking controller

$$u(t) = -k(t)y(t) \quad \text{where} \quad \dot{k}(t) = d_{\lambda}(|e(t)|)^2, \; k(0) = 1 = \lambda$$

where error $e(t) = y_{\text{ref}}(t) - y(t)$ and $y_{\text{ref}}(\cdot) = 10$.

4.4.2 Admissible funnels

The funnel boundary $\psi(\cdot)$ must be chosen from the following set

$$\mathcal{B}_1 := \left\{ \psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0} \mid \exists c > 0 : \psi(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}; [c, \infty)) \right\},$$

which allows to introduce the performance funnel

$$\mathcal{F}_\psi := \left\{ (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid |e| < \psi(t) \right\}$$

with funnel boundary $\partial \mathcal{F}_\psi(t) = \psi(t)$ for all $t \geq 0$. 
Examples 4.2. Let $T_L, T_E > 0 \text{ [s]}$ and $\Lambda \geq \lambda > 0$.

- $\psi_L: \mathbb{R}_{\geq 0} \rightarrow [\lambda, \Lambda], \quad t \mapsto \psi_L(t) := \max \{\Lambda - t/T_L, \lambda\}$
- $\psi_E: \mathbb{R}_{\geq 0} \rightarrow (\lambda, \Lambda], \quad t \mapsto \psi_E(t) := (\Lambda - \lambda) \exp\left(-t/T_E\right) + \lambda$.

4.4.3 Application: Funnel speed control of unsaturated ($\text{1MS}^\omega$)

Theorem 4.3 (Funnel speed control of unsaturated ($\text{1MS}^\omega$)). Consider the mechatronic system ($\text{1MS}^\omega$) with ($\text{1MS}^\omega$-Data) and $\dot{u}_A = \infty$ (unsaturated actuator). Then, for any funnel boundary $\psi(\cdot) \in \mathcal{B}_1$, gain scaling function $\varsigma(\cdot) \in \mathcal{B}_1$, reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ and initial value $\omega(0) = \omega_0 \in \mathbb{R}$ satisfying

$$|y_{\text{ref}}(0) - c_1\omega(0)| < \psi(0), \quad \text{(FC1-init)}$$

the funnel controller

$$u(t) = \text{sign}(g_r)k(t)e(t) \quad \text{where} \quad e(t) = y_{\text{ref}}(t) - y(t) \quad \text{and} \quad k(t) = \frac{\varsigma(t)}{\psi(t) - |e(t)|} \quad \text{(FC1)}$$

applied to ($\text{1MS}^\omega$) yields a closed-loop initial-value problem with the properties

(i) there exists a unique solution $\omega: [0, T) \rightarrow \mathbb{R}$ with maximal $T \in (0, \infty]$;

(ii) the solution $\omega(\cdot)$ does not have finite escape time, i.e. $T = \infty$;

(iii) the tracking error is uniformly bounded away from the funnel boundary, i.e.

$$\exists \varepsilon > 0 \forall t \geq 0: \quad \psi(t) - |e(t)| \geq \varepsilon;$$

(iv) gain and control action are uniformly bounded, i.e. $k(\cdot), u(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$. 

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5 Funnel speed control of unsaturated 1MS (2012/06/22)

In this lecture, we discussed an outline (which is relevant for the exam) of the proof of Theorem 4.3. In the following the full proof is presented. Since we allow for essentially bounded disturbances \( u_A(\cdot) \) and \( m_L(\cdot) \), we must apply an extended existence theory in the sense of Carathéodory (see Section D.2 in script: Definition D.35 and Theorem D.36).

Proof of Theorem 4.3. Step 1: Some preliminaries.

First, note that \( \text{sign}(g_r) = \frac{\text{sign}(\frac{c_1 k_A}{\Theta})}{g_r} \) and, for

\[
e(t) = y_{\text{ref}}(t) - y(t) \iff (a_1 + \frac{\nu_1 + \nu_2}{g_x^2}) = 0 \quad \text{and} \quad g: \mathbb{R} \to \mathbb{R}, \quad \omega \mapsto g(\omega) := \frac{f_1(\omega)}{\Theta} + \frac{f_2}{g_r},
\]

rewrite (1MS\(^\circ\)), (FC\(_1\)) in error ODE, i.e.,

\[
\begin{align*}
\dot{e}(t) &= \dot{y}_{\text{ref}}(t) - \dot{y}(t) \overset{(\text{IM}S^\circ)}{=} \dot{y}_{\text{ref}}(t) - c_1 \dot{\omega}(t) \\
&= \dot{y}_{\text{ref}}(t) + a_1 c_1 \omega(t) - \frac{c_1 k_A}{\Theta} \left( \text{sign}(g_r) k(t) e(t) + u_A(t) \right) + c_1 g(\omega(t)) + \frac{c_1 m_L(t)}{g_r} \\
&= \dot{y}_{\text{ref}}(t) + a_1 (y_{\text{ref}}(t) - e(t)) - |\gamma_0| k(t) e(t) - \gamma_0 u_A(t) + c_1 g \left( \frac{1}{c_1} (y_{\text{ref}}(t) - e(t)) \right) + \frac{c_1}{g_r} m_L(t).
\end{align*}
\]

In more compact form:

\[
e(t) = f(t, e(t)) := -\left( a_1 + |\gamma_0| k(t) \right) e(t) + a_1 y_{\text{ref}}(t) + \dot{y}_{\text{ref}}(t) - \gamma_0 u_A(t) + c_1 g \left( \frac{1}{c_1} (y_{\text{ref}}(t) - e(t)) \right) + \frac{c_1}{g_r} m_L(t) \quad \text{ (CL)}
\]

Step 2: We show Assertion (i).

In view of the Carathéodory Existence Theorem D.36, we need open state domain \( D \). \rightarrow \text{Trick:}

introduce augmented system with “time state” \( \tau \), i.e. \( \dot{\tau}(t) = 1, \tau(0) = \tau_0 = 0 \). Then, for

\[
f: \mathcal{I} \times D \to \mathbb{R}^2, \quad (t, (\tau, e)) \mapsto \begin{pmatrix}
-\left( a_1 + |\gamma_0| \psi(\tau) \right) e + a_1 y_{\text{ref}}(t) + \dot{y}_{\text{ref}}(t) \\
+ c_1 g \left( \frac{1}{c_1} (y_{\text{ref}}(t) - e(t)) \right) + \frac{c_1}{g_r} m_L(t) - \gamma_0 u_A(t)
\end{pmatrix} \quad \text{ (RHS)}
\]

where

\[
\mathcal{I} := \mathbb{R}_{\geq 0} \quad \text{ and } \quad \text{open “state domain”} \ \mathcal{D} := \{(\tau, e) \in \mathbb{R} \times \mathbb{R} \mid |e| < \psi(\theta)\}.
\]

Hence, for augmented state \( \hat{x} := (\tau, e) \), we may write unsaturated (1MS\(^\circ\)), (FC\(_1\)) in standard
form, i.e.
\[
\frac{d}{dt} \hat{x}(t) = f(t, \hat{x}(t)), \quad \hat{x}(0) = \left( y_{\text{ref}}(0) - c_1 \omega_0 \right).
\]

Note that all exogenous signals are bounded, i.e. \(u_A(\cdot), \varsigma(\cdot), y_{\text{ref}}(\cdot), m_L(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})\) and the perturbation is bounded, i.e. \(g(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})\). So, for any compact set \(\mathcal{C} \subset \mathcal{D}\), the following holds
\[
\begin{align*}
\exists M_e > 0 & \quad \forall (\tau, e) \in \mathcal{C} : \quad \|(\tau, e)\| \leq M_e \\
\exists m_e > 0 & \quad \forall (\tau, e) \in \mathcal{C} : \quad \min\{\psi(|\tau|) - |e|\} \geq m_e.
\end{align*}
\]

To apply Theorem D.36, we have to check whether \(f(\cdot, \cdot)\) satisfies the Carathéodory conditions (\(\text{car}_1\), \ldots, \(\text{car}_4\)) (see Definition D.35 on p. 119):

- \(\text{car}_1\) \(f(t, \cdot)\) is continuous on \(\mathcal{D}\) for (almost) all \(t \in \mathcal{I}\). \(\rightarrow \text{OK}\)
- \(\text{car}_2\) \(f(\cdot, (\tau, e))\) is measurable (e.g. all exogenous signals are piecewise continuous) for each fixed \((\tau, e) \in \mathcal{D}\). \(\rightarrow \text{OK}\)
- \(\text{car}_3\) \(\|f(\cdot, (\tau, e))\|\) is locally integrable on \(\mathcal{I}\) for each fixed \((\tau, e) \in \mathcal{D}\), since
\[
\|f(t, (\tau, e))\| \leq 1 + a_1 M_e + |\gamma_0| \frac{\|k\|_{\mathcal{M}e}}{M_e} + a_1 \|y_{\text{ref}}\|_{\mathcal{M}e} + \|y_{\text{ref}}\|_{\mathcal{M}e} + \|u_A\|_{\mathcal{M}e} + \|m_L\|_{\mathcal{M}e} + |\gamma_0| + \frac{c_1}{\Theta} \|g\|_{\mathcal{M}e} + |\gamma_0| + \|y_{\text{ref}}\|_{\mathcal{M}e} = : l_e < \infty. \quad \rightarrow \text{OK}
\]
- \(\text{car}_4\) First note that, since \(\psi(\cdot) \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})\) is globally Lipschitz, i.e. there exists \(L_\psi > 0\) such that \(|\psi(t_1) - \psi(t_2)| \leq L_\psi |t_1 - t_2|\) for all \(t_1, t_2 \geq 0\), the following holds for all \((\tau, e), (\tilde{\tau}, \tilde{e}) \in \mathcal{C}\):
\[
\begin{align*}
\frac{|e|}{\psi(|\tau|) - |e|} - \frac{|\tilde{e}|}{\psi(|\tilde{\tau}|) - |\tilde{e}|} & \leq \frac{1}{\psi(|\tau|) - |e|} - \frac{1}{\psi(|\tilde{\tau}|) - |\tilde{e}|} + \frac{1}{\psi(|\tilde{\tau}|) - |\tilde{e}|} |e - \tilde{e}| \\
& \leq M_e \left( \frac{\psi(|\tilde{\tau}|) - \psi(|\tau|) + |e|}{\psi(|\tau|) - |e|} - \frac{1}{\psi(|\tilde{\tau}|) - |\tilde{e}|} \right) + \frac{1}{m_e} |e - \tilde{e}| \\
& \leq M_e \left( \frac{\psi(|\tilde{\tau}|) - \psi(|\tau|) + |e|}{\psi(|\tau|) - |e|} \right) + \frac{1}{m_e} |e - \tilde{e}| \\
& \leq \frac{M_e + 1}{m_e} |e - \tilde{e}| + \frac{M_e}{m_e} L_\psi |\tau - \tilde{\tau}|. \quad (SC_1)
\end{align*}
\]

Moreover, since \(f_1(\cdot)\) and \(f_2(\cdot)\) are locally Lipschitz (due to \((1\text{MS}^\omega)-\text{Data})\), there exists \(L_g > 0\) such that the following holds for all \((t, e), (\tilde{t}, \tilde{e}) \in \mathcal{C}\)
\[
\begin{align*}
|g \left( \frac{1}{c_1} (y_{\text{ref}}(t) - e) \right) - g \left( \frac{1}{c_1} (y_{\text{ref}}(\tilde{t}) - \tilde{e}) \right)| & \leq L_g \left| \frac{1}{c_1} y_{\text{ref}}(t) - e - \frac{1}{c_1} y_{\text{ref}}(\tilde{t}) + \frac{\tilde{e}}{c_1} \right| \\
& \leq \frac{L_g}{|c_1|} |e - \tilde{e}| \quad (SC_2)
\end{align*}
\]
Hence, for each compact \( \tilde{C} \subset I \times D \) exists \( l_{\tilde{e}} > 0 \), such that for all \( (t, (\tau, e)), (\tilde{t}, (\tilde{\tau}, \tilde{e})) \in \tilde{C} \):

\[
\| f(t, (\tau, e)) - f(t, (\tilde{\tau}, \tilde{e})) \| \leq |a_1| |e - \tilde{e}| + |\gamma_0| |\varsigma(t)| \left( \frac{e}{\psi(|\tau|)} - |e| - \frac{\tilde{e}}{\psi(|\tilde{\tau}|)} \right) \\
+ |c_1| \left[ g \left( \frac{1}{c_1} (y_{\text{ref}}(t) - e) \right) - g \left( \frac{1}{c_1} (y_{\text{ref}}(t) - \tilde{e}) \right) \right] \\
\leq l_{\tilde{e}} \left\| \left( \tau, e \right) - \left( \tilde{\tau}, \tilde{e} \right) \right\|. \quad \rightarrow \text{OK}
\]

So all four Carathéodory conditions are satisfied and Theorem D.36 gives a unique solution \((\tau, e) : [0, T) \to \mathbb{R}^2\) with maximal \( T \in (0, \infty) \). Moreover, \( f(\cdot, \cdot) \) is locally essentially bounded, i.e., for any compact \( \mathfrak{A} \subset I \times D \), we have

\[
\forall (t, (\tau, e)) \in \mathfrak{A} : \| f(t, (\tau, e)) \| \leq 1 + \left( |a_1| + \left\| \varsigma \right\|_\infty \right) M_{\mathfrak{A}} + |a_1| \| y_{\text{ref}} \|_\infty + \| \dot{y}_{\text{ref}} \|_\infty \\
+ |c_1| \| g \|_\infty + \frac{|c_1|}{g_{r, \Theta}} \| m_L \|_\infty + |\gamma_0| \| u_A \|_\infty < \infty.
\]

Now, if \( T < \infty \), then Theorem D.36 also gives for any compact \( \tilde{C} \subset D \) existence of \( \tilde{t} \in [0, T) \) such that \( \hat{x}(\tilde{t}) = \left( \begin{array}{c} \tau(\tilde{t}) \\ e(\tilde{t}) \end{array} \right) \notin \tilde{C} \) (the solution leaves any compact set).

Step 3: We show a key inequality.

Note that, in view of Step 2, we have

\[
\forall t \in [0, T) : \quad |e(t)| < \psi(t) \leq \| \psi \|_\infty \quad (\Box)
\]

and so

\[
\text{for a.a. } t \in [0, T) : \quad \dot{e}(t) \leq |a_1| \| \psi \|_\infty - |\gamma_0| k(t) e(t) + |a_1| \| y_{\text{ref}} \|_\infty + \| \dot{y}_{\text{ref}} \|_\infty \\
+ \frac{|c_1|}{\Theta} \| g \|_\infty + \frac{|c_1|}{g_r, \Theta} \| m_L \|_\infty + |\gamma_0| \| u_A \|_\infty.
\]

Hence, for

\[
M := |a_1| \left( \| \psi \|_\infty + \| y_{\text{ref}} \|_\infty \right) + \| \dot{y}_{\text{ref}} \|_\infty + |\gamma_0| \| u_A \|_\infty + |c_1| \left( \| g \|_\infty + \frac{1}{g_r} \| m_L \|_\infty \right) \quad (\triangle)
\]

the following holds

\[
\text{for a.a. } t \in [0, T) : \quad -M - |\gamma_0| k(t) e(t) \leq \dot{e}(t) \leq M - |\gamma_0| k(t) e(t). \quad (K)
\]

Step 4: For \( M \) as in \( \triangle \),

\[
\varsigma := \inf_{t \geq 0} \varsigma(t) \quad \text{and} \quad \lambda := \inf_{t \geq 0} \psi(t), \quad (**)\]

---

— Page 27/41 —
we show that there exists a positive
\[ \varepsilon \leq \min \left\{ \frac{\lambda}{2}, \psi(0) - |e(0)|, \frac{|\gamma_0| \delta \lambda}{2 \left( M + \|\psi\|_{\infty} \right)} \right\} \] (***)

such that
\[ \forall t \in [0, T) : \quad \psi(t) - |e(t)| \geq \varepsilon. \]

Seeking a contradiction, assume there exists (see Fig. 11)
\[ t_1 := \min \{ t \in [0, T) \mid \psi(t) - |e(t)| < \varepsilon \} . \]

Then, by continuity of \( \psi(\cdot) - |e(\cdot)| \) on \([0, T)\), there exists
\[ t_0 := \max \{ t \in [0, t_1) \mid \psi(t) - |e(t)| = \varepsilon \} . \]

and, for \( \varepsilon > 0 \) as in (***), we have
\[ \forall t \in [t_0, t_1] : \quad |e(t)| \geq (**) \psi(t) - \varepsilon \geq (**) \lambda - \frac{\lambda}{2} = \frac{\lambda}{2} , \] (E)

Hence, sign \( e(\cdot) \) is constant on \([t_0, t_1] \subset [0, T)\). We only consider the case \( e(\cdot) > 0 \) on \([t_0, t_1]\), the other case follows analogously. Now, clearly,
\[
\text{for a.a. } t \in [t_0, t_1]: \quad \dot{e}(t) \leq M - |\gamma_0| \frac{\delta}{\psi(t) - |e(t)|} e(t) \\
\quad \leq (**) (E) \leq M - |\gamma_0| \frac{\delta \lambda}{2\varepsilon} \leq -\|\dot{\psi}\|_{\infty}
\]

and integration yields
\[
\forall t \in [t_0, t_1]: \quad e(t) - e(t_0) = \int_{t_0}^{t} \dot{e}(\tau) \, d\tau \leq -\|\psi\|_{\infty} (t - t_0) .
\]

This, with the properties of \( \psi(\cdot) \in \mathcal{B}_1 \) (locally Lipschitz), i.e.
\[
\forall t \geq t_0 : \quad \psi(t) \geq \psi(t_0) - \|\dot{\psi}\|_{\infty} (t - t_0) ,
\]
gives
\[ \forall t \in [t_0, t_1] : \ e(t) - e(t_0) \leq -\|\dot{\psi}\|_\infty (t - t_0) \leq \psi(t) - \psi(t_0), \]
with which we arrive at the contradiction
\[ \varepsilon = \psi(t_0) - e(t_0) \leq \psi(t) - e(t) \leq \varepsilon. \]
(by initial assumption)
\[ \Rightarrow \psi(t) - |e(t)| \geq \varepsilon \text{ for all } t \in [0, T], \] which completes Step 4.

Step 5: We show that Assertions (ii), (iii) and (iv) hold true.
First, we show Assertion (ii). For \( \varepsilon > 0 \) as in \((\star\star\star)\), define the compact set
\[ \tilde{C} := \{(t, e) \in [0, T] \times \mathbb{R} \mid |e| \leq \psi(t) - \varepsilon\} \subset D, \]
where \( D = \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid |e| < \psi(t)\} \) as in Step 2. Note that, if \( T < \infty \), \( \tilde{C} \) is a compact subset of \( D \). Moreover,
\[ \forall t \in [0, T] : \ \left(\begin{array}{c} t \\ e(t) \end{array}\right) \in \tilde{C}, \]
therefore the whole graph of the solution “evolves” in \( \tilde{C} \). But this, in view of Theorem D.36, contradicts maximality of \( T \). Hence \( T = \infty \).
Now, Assertion (iii), i.e.
\[ \forall t \geq 0 : \ \psi(t) - |e(t)| \geq \varepsilon, \]
follows from Step 4, which also implies, for any \( \varsigma(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \), that \( k(\cdot) = \frac{\varsigma(\cdot)}{\psi(\cdot) - |e(\cdot)|} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \). This with \((\square)\) implies
\[ u(\cdot) = k(\cdot)e(\cdot) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \]
which shows Assertion (iv) and completes the proof. \( \square \)
6 Funnel speed control of saturated 1MS (2012/06/29, part 1)

Now, we will apply funnel control for speed control of \((1MS^\omega)\) with actuator saturation.

**Theorem 6.1** (Funnel speed control of saturated \((1MS^\omega)\)). Consider the mechatronic system \((1MS^\omega)\) with \((1MS^\omega)-\text{Data})\). If actuator saturation satisfies

\[
\hat{u}_A \geq \frac{\Theta}{|c_1| k_A} (\|\dot{\psi}\|_\infty + M + \delta), \quad \delta > 0 \text{ (arbitrary)} \quad \text{(FC1-feas)}
\]

where

\[
M := \frac{\nu_1 + \frac{\nu_2}{\Theta}}{\Theta} \left( \|\dot{\psi}\|_\infty + \|y_{\text{ref}}\|_\infty \right) + \|\dot{y}_{\text{ref}}\|_\infty + \left| \frac{|c_1|}{\Theta} \right| \left( \|f_1\|_\infty + \frac{\|f_2\|_\infty + \|m_L\|_\infty}{|g_r|} \right),
\]

then, for any funnel boundary \(\psi(\cdot) \in \mathcal{B}_1\), gain scaling function \(\varsigma(\cdot) \in \mathcal{B}_1\), reference signal \(y_{\text{ref}}(\cdot) \in \mathcal{W}_1^1(\mathbb{R}_{\geq 0}; \mathbb{R})\) and initial value \(\omega(0) = \omega_0 \in \mathbb{R}\) satisfying \(|y_{\text{ref}}(0) - c_1 \omega(0)| < \psi(0)\), the funnel controller

\[
u(t) = \text{sign}(g_r) k(t) e(t) \quad \text{where} \quad e(t) = y_{\text{ref}}(t) - y(t) \quad \text{and} \quad k(t) = \frac{\varsigma(t)}{\psi(t) - |e(t)|} \quad \text{(FC1)}
\]

applied to \((1MS^\omega)\) yields a closed-loop initial-value problem with the following properties:

(i) there exists a unique solution \(\omega : [0, T) \to \mathbb{R}\) with maximal \(T \in (0, \infty]\);

(ii) the solution \(\omega(\cdot)\) does not have finite escape time, i.e. \(T = \infty\);

(iii) the tracking error is uniformly bounded away from the funnel boundary, i.e.

\[
\exists \varepsilon > 0 \quad \forall t \geq 0 : \quad \psi(t) - |e(t)| \geq \varepsilon;
\]

(iv) gain and control action are uniformly bounded, i.e. \(k(\cdot), u(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})\).

Clearly, to evaluate the feasibility condition \((\text{FC1-feas})\) rough system knowledge is necessary. Moreover, since the feasibility condition is derived by worst case analysis, the upper bound on the actuator saturation \(\hat{u}_A\) may be a very conservative and so unrealistic bound.

**Proof of Theorem 6.1.**

Step 1: Some preliminaries.

We rewrite \((1MS^\omega)\), \((\text{FC1})\) in error ODE:

\[
\dot{e}(t) = -a_1 (y_{\text{ref}}(t) + e(t)) + \dot{y}_{\text{ref}}(t) - \frac{c_1 k_A}{\Theta} \text{sat}_{\hat{u}_A} \left( \text{sign}(g_r) k(t) e(t) + u_A(t) \right) \\
+ \frac{c_1}{\Theta} \left( f_1 \left( \frac{1}{c_1} (y_{\text{ref}}(t) - e(t)) \right) + \frac{1}{g_r} f_2 \left( \frac{1}{c_1 g_r} (y_{\text{ref}}(t) - e(t)) \right) + \frac{m_L(t)}{g_r \Theta} \right) \quad \text{(CL)}
\]

Step 2: We show Assertion (ii).

... skipped; similar as in unsaturated case (see Proof of Theorem 4.3).
Step 3: We show a key inequality. From Step 2, we have

$$\forall t \in [0, T) : \quad |e(t)| < \psi(t) \leq \|\psi\|_{\infty}$$

and so

$$\forall t \in [0, T) : \quad \dot{e}(t) \leq |a_1| (\|y_{\text{ref}}\|_{\infty} + \|\psi\|_{\infty}) + \|y_{\text{ref}}\|_{\infty} + \frac{|c_1|}{\Theta} \left[ \|f_1\|_{\infty} + \frac{1}{|g_r|} (\|f_2\|_{\infty} + \|m_L\|_{\infty}) \right] - \frac{c_1 k_A}{\Theta} \left( \text{sign}(g_r) k(t) e(t) + u_A(t) \right)$$

Hence,

$$\forall t \in [0, T) : \quad -M - |\gamma_0| \text{sat}_{u_A} \left( k(t) e(t) - \|u_A\|_{\infty} \right) \leq \dot{e}(t) \leq M - |\gamma_0| \text{sat}_{u_A} \left( k(t) e(t) - \|u_A\|_{\infty} \right)$$

(KI)

Step 4: For $M$ as above and

$$\zeta := \inf_{t \geq 0} \zeta(t) \quad \text{and} \quad \lambda := \inf_{t \geq 0} \psi(t),$$

we show that there exists a positive

$$\varepsilon \leq \min \left\{ \frac{\lambda}{2}, \psi(0) - |e(0)|, \frac{\zeta \lambda}{2(\hat{u}_A + \|u_A\|_{\infty})} \right\}$$

(***)

such that

$$\forall t \in [0, T) : \quad \psi(t) - |e(t)| \geq \varepsilon.$$
Now, identical arguments as in the proof of Theorem 4.3 show the claim.

Step 5: We show that Assertions (ii), (iii) and (iv) hold true.
Again identical arguments as in the proof of Theorem 4.3 show the claim.
7 Internal model principle
(2012/06/29 & 2012/07/13)

7.1 Motivation (2012/06/29, part 2)

With funnel control, we can achieve tracking with prescribed asymptotic accuracy, but the tracking error does not necessarily tend to zero: We are not allowed to choose a funnel boundary which tends to zero (recall admissible boundaries in set \( \mathcal{B}_1 \) on p. 65), hence we cannot guarantee \( \lim_{t \to \infty} e(t) = 0 \).

Following the internal model principle (as stated by Wonham in the 80s):

“Every good regulator must incorporate a model of the outside world being capable to reduplicate the dynamic structure of the exogenous signals which the regulator is required to process [3, p. 210],”

we are able to significantly improve the tracking performance.

What is the standard solution in industry to achieve steady state accuracy for constant references and/or disturbances?

It is a PI controller, which is actually the most simple internal model (capable to reduplicate constant signals):

- transfer function: \( F_{PI}(s) = k_P + \frac{k_I}{s} = \frac{k_P s + k_I}{s} \), \( k_I, k_P > 0 \)
- relative degree: \( n = 1, m = 1 \implies r_{PI} = n - m = 0 \)
- high frequency gain: \( \gamma_{0,PI} = \lim_{s \to \infty} s^{r} F_{PI}(s) = \lim_{s \to \infty} \left( k_P + \frac{k_I}{s} \right) = k_P \)
- minimum-phase, since for all \( s \in \mathbb{C}_{\geq 0} : sk_P + k_I \neq 0 \) \( (s = -\frac{k_I}{k_P} < 0) \)

Now, additionally, consider a linear system given in the frequency domain:

- transfer function: \( F_S(s) = \frac{\gamma_0 N(s)}{D(s)} \), \( N, D \) monic; \( N \) Hurwitz
- relative degree: \( r = \deg(N) - \deg D = 1 \)
- high frequency gain: \( \gamma_0 \) known
- minimum-phase, since \( N \) is Hurwitz.

Analyzing the serial interconnection of PI controller \( F_{PI}(s) \) and \( F_S(s) \) yields an augmented system (see Fig. 12):

![Figure 12: Serial interconnection of PI controller \( F_{PI}(s) \) and system \( F_S(s) \): the augmented system with “new” control input \( v(t) \) controlled by e.g. funnel controller \( FC_1 \).](image)
Non-identifier based adaptive control in mechatronics

Transfer function:
\[ F_{PI}(s)F(s) = \frac{k_p s + k_I}{s} \cdot \gamma_0 \frac{N(s)}{D(s)} = k_p \frac{s + \frac{k_I}{k_p}}{s} \cdot \frac{N(s)}{D(s)} \]

- relative degree: \( r = 1 \)
- high frequency gain: \( \tilde{\gamma}_0 = k_p \gamma_0 \)
- minimum-phase, since \( s + \frac{k_I}{k_p} \) and \( N(s) \) are Hurwitz \( \implies \) the augmented system is minimum-phase.

Concluding, the augmented system is again element of system class \( S_{\text{lin}}^1 \) (see Definition 2.19) and hence any high-gain adaptive controller (e.g. funnel controller (FC1)) for relative-degree-one systems is still applicable for the serial interconnection.

### 7.2 Internal model principle

For non-identifier based adaptive control:

- What is an “internal model”?
- Which “exogenous signals” can it reduplicate (admissible function class \( Y_{\text{ref}} \))?
- How is an “internal model” designed (realized)?
- Does “augmented system” has same properties as “system” (relative degree one, known sign of high-frequency gain, minimum-phase)?
- Can we achieve asymptotic tracking, i.e. \( \lim_{t \to \infty} e(t) = 0 \) (for linear systems)?
- Application example: PI-funnel control for speed control of \( (1 MS\omega) \)

#### 7.2.1 What is an internal model?
- dynamical linear system
- purpose to “reduplicate” a certain class of (reference and/or disturbance) signals

In the following, we only consider the case to reduplicate reference signals \( y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R} \) (the case to reduplicate certain disturbance signals can be treated analogously). An internal model is represented by an linear ordinary differential equation of \( p \) order, \( p \in \mathbb{N} \):

\[
y_{\text{ref}}^{(p)}(t) + \hat{a}_{p-1} \cdot y_{\text{ref}}^{(p-1)}(t) + \ldots + \hat{a}_1 \cdot y_{\text{ref}}(t) + \hat{a}_0 \cdot y_{\text{ref}}(0) = 0, \quad (y_{\text{ref}}^{(p-1)}(0), \ldots, y_{\text{ref}}(0)) \in \mathbb{R}^p \quad (*)
\]

with characteristic polynomial:

\[ D_{IM}(s) = s^p + \hat{a}_{p-1} \cdot s^{p-1} + \ldots + \hat{a}_1 \cdot s + \hat{a}_0 \in \mathbb{R}[s], \text{i.e. } \hat{a}_i \in \mathbb{R} \text{ for all } i \in \{1, \ldots, p\} \]
From standard linear control theory and experience, we know that a proportional controller can only exhibit a non-zero control action as soon as the control error is non-zero to attenuate e.g. constant disturbances. To circumvent this drawback, typically, a PI controller is introduced which allows to generate a non-zero control action even if the control error is zero! In a more general setup, the same task has an internal model introduced in (⋆) for a wider “exogenous” signal class.

7.2.2 Which “exogenous signals” can a linear internal model reduplicate?

- all signals, which are the solutions of the linear ODE (⋆), where \( y_{\text{ref}}(\cdot) \in C^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \)
- since we are interested in asymptotic tracking, \( \lim_{t \to \infty} e(t) = 0 \), we do not consider references which tend to zero: in the following, let \( \lim_{t \to \infty} y_{\text{ref}}(t) \neq 0 \).
- the linear system (⋆) is assumed to be not asymptotically stable, more precisely, we only consider \( D_{IM} \) with roots in the left complex half-plane and on the imaginary axis, i.e.,

\[
\mathcal{Y}_{\text{ref}} := \left\{ y_{\text{ref}}(\cdot) \in C^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \mid \begin{array}{c}
D_{IM} \left( \frac{d}{dt} \right) y_{\text{ref}}(t) = 0, \\
\text{corresponds to (⋆)} \\
\end{array} D_{IM} \in \mathbb{R}[s], \text{ monic with } \\
\{ s_0 \in \mathbb{C} \mid D_{IM}(s_0) = 0 \} \subset \mathbb{C}_{\geq 0} \right\}.
\]

7.2.3 How is an “internal model” designed (realized)?

Goal: Find state space realization of the internal model.

Procedure:

- given \( y_{\text{ref}}(\cdot) \in \mathcal{Y}_{\text{ref}} \) (e.g. ramps, sinusoidal, constants or linear combinations thereof), find the Laplace transform:

\[
y_{\text{ref}}: t \to \mathbb{R} \quad \longrightarrow \quad F_{\text{ref}}(s) = \frac{\ldots}{D_{IM}(s)} \Rightarrow D_{IM}(s)
\]

- design of internal model in frequency domain:

\[
F_{IM}(s) = \frac{u(s)}{v(s)} = \frac{N_{IM}(s)}{D_{IM}(s)}
\]

where \( N_{IM}(s) \) such that

\[
\begin{array}{l}
\left\{ \begin{array}{l}
\gamma_{IM}^0 > 0 \\
\text{minimum phase}
\end{array} \right\}
\end{array}
\]

choice of \( N_{IM}(s) \) as follows:

\[
\begin{array}{l}
p := \text{deg}(D_{IM}) = \text{deg}(N_{IM}) \Rightarrow N_{IM}(s) = c_p s^p + \ldots + c_0, \; c_p \neq 0 \Rightarrow r_{IM} = 0 \\
\lim_{s \to \infty} s^0 F_{IM}(s) = \gamma_{IM}^0 > 0 \Rightarrow c_p = \gamma_{IM}^0 > 0 \\
N_{IM}(s) \text{ Hurwitz } \Rightarrow F_{IM}(s) \text{ is minim-phase}
\end{array}
\]

- design of a minimal (state space) realization of \( F_{IM}(s) \):

\[
\begin{array}{l}
\dot{x}(t) = \hat{A} x(t) + \hat{b} v(t), \quad \hat{x}(0) = \hat{x}_0 \in \mathbb{R}^p \\
u(t) = \tilde{c}^T \hat{x} + \gamma_{IM}^0 v(t)
\end{array}
\]

(IMSS)
with “new” (augmented) control input $v(\cdot)$. To choose matrix $\hat{A}$, vectors $\hat{b}, \hat{c}^\top$ and feedthrough $\gamma_{0}^{IM}$ adequately, rewrite

$$F_{IM}(s) = \frac{N_{IM}(s)}{D_{IM}(s)} = \hat{N}_{IM}(s)\gamma_{0}^{IM}$$

where $N_{IM}(s) = \hat{N}_{IM} + \gamma_{0}^{IM}D_{IM}(s)$

and $\hat{N}_{IM}(s) = \hat{c}_{p-1} \cdot s^{p-1} + \cdots + \hat{c}_{1} \cdot s + \hat{c}_{0}$, $\hat{c}_{i} > 0$ for all $i \in \{1, \ldots, p-1\}$

and then choose

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
-\hat{a}_{0} & -\hat{a}_{1} & \cdots & -\hat{a}_{p-2} & -\hat{a}_{p-1} \end{bmatrix} \in \mathbb{R}^{p \times p}, \quad \hat{b} = \begin{bmatrix} 0 \\
\vdots \\
0 \\
1 \end{bmatrix} \in \mathbb{R}^{p}, \quad \hat{c} = \begin{bmatrix} \hat{c}_{0} \\
\vdots \\
\hat{c}_{p-1} \end{bmatrix} \in \mathbb{R}^{p}$$

according to the controllable canonical form.

### 7.2.4 Does the “augmented system” has same properties as “system”?

The serial interconnection of $(IM_{SS})$ and linear system

$$\begin{align*}
\dot{x} &= Ax + bu, \quad x(0) = x_{0} \in \mathbb{R}^{n} \\
y &= c^\top x 
\end{align*}$$

$(SYS)$

is given by

$$\begin{align*}
\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} A & b \hat{c} \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} b_{s}^{IM} \\ \hat{b} \end{bmatrix} v(t) \\
y &= (c^\top, 0_{p}^\top) x_{s}
\end{align*}$$

$(IM+SYS)$

with properties (see Lemma 2.38):

- if system $(SYS)$ has relative degree $1 \leq r \leq n$, then so the serial interconnection $(IM+SYS)$.
- the high-frequency gain of $(IM+SYS)$ has same sign as the high-frequency gain of $(SYS)$
- if $(SYS)$ is minimum phase, then so $(IM+SYS) \Rightarrow (IM+SYS) \in S_{1}^{\text{lin}}$
7.2.5 Can we achieve asymptotic tracking, i.e. \( \lim_{t \to \infty} e(t) = 0 \)?

Can we achieve asymptotic tracking, i.e. \( \lim_{t \to \infty} e(t) = 0 \) for some \( y_{\text{ref}}(\cdot) \in \mathcal{Y}_{\text{ref}} \), if we apply the high gain adaptive (tracking) controller

\[
v(t) = \text{sign}(\gamma_0) k(t) e(t), \quad \text{where} \quad e(t) = y_{\text{ref}}(t) - y(t)
\]

\[
k(t) = q_1 |e(t)|^{q_2}, \quad k(0) = k_1, \quad q_1 > 0, \quad q_2 \geq 1
\]

(HGAC)

to linear system \((\text{IM+SYS})\)? \(\Rightarrow\) Yes, see Theorem 2.39.

Sketch of proof of Theorem 2.39: Consider \( \dot{w} = A_s w \) which has a global solution \( w(\cdot) \) on \( \mathbb{R}_{\geq 0} \) and \( w(\cdot) \in C^{\infty} \). In view of Lemma 5.1.2 in [2], there exists \( w_0^{\text{ref}} \in \mathbb{R}^{n+p} \), such that

\[
\begin{align*}
\dot{w}(t) &= A_s w(t), \quad w(0) = w_0^{\text{ref}} \in \mathbb{R}^{n+p} \\
y_{\text{ref}}(t) &= c_s^{\top} w(t).
\end{align*}
\]

allows to rewrite the tracking problem of \((\text{HGAC}), (\text{IM+SYS})\) in \( x_e(\cdot) \), i.e.

\[
\begin{align*}
\dot{x}_e(t) &= \dot{w}(t) - \dot{x}_e(t) = A_s x_e(t) - A_s x_e(t) - b_s v(t), \quad x_e(0) = w_0^{\text{ref}} - x_0^e \in \mathbb{R}^{n+p} \\
e(t) &= c_s^{\top} x_e(t) \quad \text{(output is now the error)}.
\end{align*}
\]

In view of Lemma 2.38, system \((\triangle)\) is element of class \( S_1^{\text{lin}} (r = 1, \text{known high-frequency gain, minimum phase}) \), hence Theorem 2.21 yields

\[
\lim_{t \to \infty} x_e(t) = 0 \quad \Rightarrow \quad \lim_{t \to \infty} e(t) = \lim_{t \to \infty} c_s^{\top} x_e(t) = 0.
\]

By introducing the virtual system \((\#)\) (which corresponds to the serial interconnection \((\text{IM+SYS})\) for \( v(\cdot) = 0 \)), we were able to express the tracking problem \((\text{HGAC}), (\text{IM+SYS})\) as stabilization problem, i.e. that the output (here: \( e(\cdot) \)) asymptotically tends to zero.

7.3 Funnel control with internal model

- use of (linear) internal models admissible (see Lemma 2.40), if
  - relative degree \( r_{\text{IM}} = 0 \)
  - positive high-frequency gain \( \gamma_0^{\text{IM}} > 0 \)
  - controllable and observable \( \Rightarrow \) minimum-phase
Theorem 7.1 (PI-Funnel speed control of unsaturated 1MS)

Consider the mechatronic system (1MSω) with (1MSω-Data) and $\dot{u}_A = \infty$. For any funnel boundary $\psi(\cdot) \in \mathcal{B}_1$, gain scaling function $\zeta(\cdot) \in \mathcal{B}_1$, reference signal $y_{\text{ref}}(\cdot) \in \mathcal{W}^{1,\infty}(\mathbb{R}_0; \mathbb{R})$ and initial value $\omega(0) = \omega \in \mathbb{R}$ satisfying $|y_{\text{ref}}(0) - c_1 \omega(0)| < \psi(0)$, the PI-funnel controller

$$
\begin{align*}
\dot{x}_I(t) &= k(t) e(t), \\
u(t) &= \text{sign}(g_r)(k(t) e(t) + k_I x_I(t))
\end{align*}
$$

where $e(t) = y_{\text{ref}}(t) - \hat{y}(t)$ and $k(t) = \frac{\zeta(t)}{\psi(t) - |e(t)|}$

applied to (1MSω) with known $\text{sign}(g_r)$ yields a closed-loop initial-value problem with the following properties:

(i) there exists a unique solution $(\omega, x_I) : [0, T) \to \mathbb{R} \times \mathbb{R}$ with maximal $T \in (0, \infty]$;

(ii) the solution $(\omega(\cdot), x_I(\cdot))$ does not have finite escape time, i.e. $T = \infty$;

(iii) the tracking error is uniformly bounded away from the funnel boundary, i.e.

$$
\exists \varepsilon > 0 \forall t \geq 0 : \quad \psi(t) - |e(t)| \geq \varepsilon;
$$

(iv) gain, control action and integral state are uniformly bounded, i.e. $k(\cdot), u(\cdot), x_I(\cdot) \in \mathcal{L}^{\infty}(\mathbb{R}_0; \mathbb{R})$.

(v) if steady state is reached, i.e. $\lim_{t \to \infty} \hat{\omega}(t) = 0$, $\dot{x}_I(t) = 0$, and $\lim_{t \to \infty} \hat{y}_{\text{ref}}(t) = 0$, then asymptotic tracking is achieved, i.e. $\lim_{t \to \infty} e(t) = 0$ (and $\lim_{t \to \infty} \hat{e}(t) = 0$).

Proof of Theorem 7.1. Introduce

$$
g : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}, \quad (t, \omega) \mapsto g(t, \omega) := -\frac{f_1(\omega)}{\Theta} - \frac{m_I(t) + f_2 \left( \frac{\omega}{\Theta} \right)}{g_r \Theta} + \frac{k_A}{\Theta} u_A(t),$$

$$a_1 := -\frac{\nu_1 + \frac{\nu_2}{\Theta}}{\Theta}.$$
\[ \dot{x}_I(t) = v(t), \quad x_I(0) = 0 \]
\[ u(t) = v(t) + k_I x_I(t), \quad k_I > 0 \}

(PI)

In view of \((1MS^\omega\text{-Data})\), note that \(g(\cdot, \cdot) \in L^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}; \mathbb{R})\). Now, consider the serial interconnection of (PI) and \((1MS^\omega)\) (unsaturated: \(\hat{u}_A = \infty\)):

\[ \dot{\omega}(t) = a_1 \omega(t) + \frac{k_A}{\Theta} k_I x_I(t) + \frac{k_A}{\Theta} v(t) + g(t, \omega(t)), \quad \omega(0) = \omega_0 \]
\[ \dot{x}_I(t) = v(t), \quad x_I(0) = 0 \]
\[ y(t) = (c_1, 0) \begin{pmatrix} \omega(t) \\ x_I(t) \end{pmatrix} \]

or, in a more compact form,

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} \omega(t) \\ x_I(t) \end{pmatrix} &= \begin{pmatrix} a_1 & \frac{k_A}{\Theta} k_I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega(t) \\ x_I(t) \end{pmatrix} + \begin{pmatrix} \frac{k_A}{\Theta} \\ 1 \end{pmatrix} v(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} g(t, \omega(t)), \\
y(t) &= (c_1, 0) \begin{pmatrix} \omega(t) \\ x_I(t) \end{pmatrix} 
\end{align*}
\]

Checking properties of system class \(S_1\):

- \(\gamma_0 := c^T b = c_1 \frac{k_A}{\Theta} \neq 0 \quad \Rightarrow \quad r = 1\)
- \(\text{sign}(\gamma_0) = \text{sign}(c^T b) = \text{sign}(g_r)\) where \(\text{sign}(g_r)\) is known
- minimum-phase property?

\[
\det \begin{bmatrix} s - a_1 & \frac{k_A}{\Theta} k_I & \frac{k_A}{\Theta} \\ 0 & s & 1 \\ c_1 & 0 & 0 \end{bmatrix} = c_1 (-1)^{3+1} \det \begin{bmatrix} -\frac{k_A}{\Theta} k_I & \frac{k_A}{\Theta} \\ 0 & s & 1 \end{bmatrix}
\]

\[
= -c_1 \frac{k_A}{\Theta} (s + k_I) \neq 0 \quad \forall s \in \mathbb{C}_{\geq 0}
\]

\(\Rightarrow\) minimum phase.

- \(g(\cdot, \cdot) \in L^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}; \mathbb{R}_{\geq 0})\) (disturbances & perturbation is bounded)

Hence, application of funnel controller

\[ v(t) = k(t) e(t) \quad \text{where} \quad k(t) = \frac{\zeta(t)}{\psi(t) - |e(t)|} \]

is admissible (see Theorem 2.33). Moreover, note that

\[
\lim_{t \to \infty} (\dot{\omega}(t), \dot{x}_I(t)) = 0 \quad \Rightarrow \quad \lim_{t \to \infty} \left( k(t) e(t) \right) = 0
\]

which shows Assertion (v).
7.3.2 Implementation: PI-Funnel speed control of (1MS<sup>ω</sup>)

- speed controller #1: Standard PI controller

\[ u(t) = k_P e(t) + k_I \int_0^t e(\tau) \, d\tau \]

- speed controller #2: PI-Funnel controller (PI-FC<sub>1</sub>)

\[ u(t) = k(t) e(t) + k_I \int_0^t k(\tau) e(\tau) \, d\tau \quad \text{where} \quad k(t) = \frac{\varsigma(t)}{\psi(t) - \left| e(t) \right|} \]

Figure 13: Implementation: Measurement results for PI-Funnel controller (PI-FC<sub>1</sub>) and standard PI controller
References

